Suggested solution of HW6

Ch8 11 Let $g(z) = M^{-1}f(Rz)$ on $\mathbb{D}$, then $|g(z)| \leq 1$. Apply Schwarz lemma on

$$h(z) = \frac{g(z) - g(0)}{1 - \overline{g(0)}g(z)}.$$

We have

$$\left|\frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}\right| \leq |w|.$$

Putting $w = R^{-1}z$, we have

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leq \frac{|z|}{MR}.$$

Ch8 14 If φ is conformal mapping from $\mathbb{H}$ to $\mathbb{D}$, $\varphi \circ \phi$ is a conformal map from $\mathbb{D}$ to $\mathbb{D}$, where $\phi(z) = i(1-z)(1+z)^{-1}$.

Thus,

$$\varphi \circ \phi(z) = e^{i\theta}\frac{z + a}{1 + \bar{a}z}$$

for some $\theta \in \mathbb{R}$ and $|a| < 1$. Therefore,

$$\begin{aligned}\varphi(w) &= e^{i\theta}\frac{\phi^{-1}(w) + a}{1 + \bar{a}\phi^{-1}(w)} \\ &= e^{i\theta}\frac{w + \beta}{w - \bar{\beta}}\frac{a - 1}{1 - \bar{a}},\end{aligned}$$

where $\beta = i(a+1)(a-1)^{-1}$. Since $|a-1| = |1-\bar{a}|$, $e^{i\theta}(a-1)(1-\bar{a})^{-1} = e^{i\sigma}$ for some $\sigma \in \mathbb{R}$.

Ch9 Q2 Since poles and zeros are isolated, we may assume that there are no zeros or poles on the boundary of lattice by slight translation. By integrating $\frac{zf'(z)}{f(z)}$ over the fundamental parallelogram, we have

$$\sum_{i=1}^{r} a_i - b_i = \frac{1}{2\pi i}\sum_{n=1}^{4}\oint_{P_n}\frac{zf'(z)}{f(z)}\,dz,$$

where $P = P_1 + P_2 + P_3 + P_4$ denotes the fundamental parallelogram and oriented anticlockwisely ($0 \to \omega_1 \to \omega_1 + \omega_2 \to \omega_2 \to 0$).

Since f is doubly periodic, $\dfrac{f'(z+\omega_k)}{f(z+\omega_k)} = \dfrac{f'(z)}{f(z)}$.

$$\int_{P_3}\frac{zf'(z)}{f(z)}\,dz = -\int_{P_1}\frac{zf'(z)}{f(z)}\,dz - \omega_2\int_{P_1}\frac{f'(z)}{f(z)}\,dz.$$

Similarly,

$$\int_{P_4}\frac{zf'(z)}{f(z)}\,dz = -\int_{P_2}\frac{zf'(z)}{f(z)}\,dz - \omega_1\int_{P_2}\frac{f'(z)}{f(z)}\,dz.$$

Therefore,

$$\frac{1}{2\pi i}\sum_{n=1}^{4}\oint_{P_n}\frac{zf'(z)}{f(z)}\,dz = a\omega_1 + b\omega_2$$

for some $a, b \in \mathbb{Z}$ as $\int_{P_i} f'f^{-1}dz \in 2\pi i\mathbb{Z}$ by periodicity.

Ch9 Q3: Since $|n+m\tau|^2 \le n^2 + m^2|\tau|^2$ for all n, m.

$$\begin{aligned}
\sum_{n,m=-\infty}^{\infty}\frac{1}{|n+m\tau|^2} &\ge \sum_{n,m=-\infty}^{\infty}\frac{1}{n^2+m^2|\tau|^2}\\
&\ge \sum_{(m,n)\in\mathbb{Z}^2}\int_{[m-1,m]\times[n-1,n]}\frac{1}{x^2+y^2|\tau|^2}dxdy\\
&= |\tau|^{-1}\int_{\mathbb{R}^2}\frac{1}{x^2+y^2}dxdy = +\infty.
\end{aligned}$$

In fact, we only need to estimate $\sum(n^2+m^2)^{-1}$.

$$\sum_{4\le n^2+m^2\le R^2}\frac{1}{n^2+m^2} \le \int_{A(2,R)}\frac{1}{(x-1)^2+(y-1)^2}dxdy = 2\pi\log R + O(1),$$

$$\sum_{1\le n^2+m^2\le R^2}\frac{1}{n^2+m^2} \ge \int_{B(R/\sqrt{2})}\frac{1}{x^2+y^2}dxdy = 2\pi\log R + O(1).$$

ch9 Q4

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]\\
&= \frac{1}{z^2} + \sum_{0<|\omega|<R}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] + \sum_{|\omega|\ge R}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right].
\end{aligned}$$

where R is to be chosen.

$$\begin{aligned}
|\wp_R(z+1) - \wp_R(z)| &= \left|\sum_{|w|<R}\left[\frac{1}{(z+1+\omega)^2} - \frac{1}{(z+\omega)^2}\right]\right|\\
&\le \left|\sum_{R-1\le|\omega|<R+1}\frac{1}{(z+w)^2}\right|.
\end{aligned}$$

Now we estimate N, the number of ω in which $|\omega| \in [R-1, R+1)$. Without loss of generality, we may assume $|\tau| > 1$. Then the collection of balls, $\{B(\omega, 1/2)\}_{|\omega|\in[R-1,R+1)}$ are disjoint and contained in $A(R-2, R+2)$. Hence,

$$N2^{-2}\pi \le \left[(R+2)^2 - (R-2)^2\right]\pi.$$

That is $N \le 32R$. So if $R > |z| + 1$,

$$|\wp_R(z+1) - \wp_R(z)| \le \frac{32R}{(R-1-|z|)^2}.$$

On the other hand,

$$\begin{aligned}
\left|\sum_{|\omega|\geq R}\left[\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right]\right| &= \left|\sum_{|\omega|\geq R}\frac{z(z+2\omega)}{(z+w)^2w^2}\right| \\
&\leq |z|\left|\sum_{|\omega|\geq R}\frac{|z|+2|\omega|}{|\omega|^2(|\omega|-|z|)^2}\right| \\
&\leq \sum_{|\omega|\geq R}\frac{|z|^2}{|\omega|^2(|\omega|-|z|)^2}+\sum_{|\omega|\geq R}\frac{2|z||\omega|}{|\omega|^2(|\omega|-|z|)^2}.
\end{aligned}$$

Using similiar estimate on the number of ω, we can split the above sum into the following form.

$$\begin{aligned}
\sum_{|\omega|\geq R}\frac{|z|^2}{|\omega|^2(|\omega|-|z|)^2} &= \sum_{i=0}^{\infty}\sum_{R+i<|\omega|<R+i+1}\frac{|z|^2}{|\omega|^2(|\omega|-|z|)^2} \\
&\leq \sum_{i=0}^{\infty}\frac{32|z|^2}{(R+i)(R+i-|z|)^2}
\end{aligned}$$

and

$$\sum_{|\omega|\geq R}\frac{2|z||\omega|}{|\omega|^2(|\omega|-|z|)^2}\leq\sum_{i=0}^{\infty}\frac{32|z|}{(R+i-|z|)^2}.$$

Here I have used the estimate $|\{R+i<|\omega|<R+i+1\}|\leq 32(R+i)$ which follows from previous estimate.

Therefore, there is some universal constant $C>0$ so that if $R>2|z|+1$,

$$\begin{aligned}
\left|\sum_{|\omega|\geq R}\left[\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right]\right| &\leq C|z|^2\sum_{i=0}^{\infty}\frac{1}{(R+i)^3}+C|z|\sum_{i=0}^{\infty}\frac{1}{(R+i)^2} \\
&\leq \bar{C}|z|^2R^{-2}+\bar{C}|z|R^{-1}
\end{aligned}$$

where $\bar{C}$ is some universal constant. To summarize, we can find $\bar{C}>0$ so that if $R>2|z|+1$,

$$|\wp(z+1)-\wp(z)|\leq\bar{C}R^{-1}+\bar{C}|z|^2R^{-2}+\bar{C}|z|R^{-1}.$$

Letting $R\to\infty$ to conclude our result.

Ch9 Q5 (a) It suffices to show that it is of order 2. As all zeros are simple and

$$\sum_{(n,m)\in\mathbb{Z}^2,(n,m)\neq(0,0)}\frac{1}{|n+m\tau|^2}=+\infty.$$

It remains to show that the order is bounded above by 2. WLOG, we may assume $|\tau|>1$.

If $|z| < 1/2$,

$$|E_2(z)| = |(1-z)e^{z+z^2/2}| = |e^{-\sum_{n=3}^{\infty} z^n/n}| \le \exp\left(c|z|^3\right).$$

Therefore, we split the product into two parts,

$$|\sigma(z)| = |z| \prod_{w\in\Lambda, |w|\le 2|z|} |E_2(z/w)| \cdot \prod_{w\in\Lambda, |w|>2|z|} |E_2(z/w)|.$$

We estimate the two product when $|z| = r \to \infty$. Using the counting estimate as demonstrated in Q4,

$$\begin{aligned}
\log \prod_{w\in\Lambda, |w|>2|z|} |E_2(z/w)| &\le c \sum_{w\in\Lambda, |w|>2|z|} \left|\frac{z}{w}\right|^3 \\
&= c \sum_{k=1}^{\infty} \sum_{2^{k+1}|z|\ge|w|>2^k|z|} \left|\frac{z}{w}\right|^3 \\
&\le \sum_{k=1}^{\infty} \frac{C2^{2k+2}|z|^2}{2^{3k}} = \tilde{C}r^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\log \prod_{w\in\Lambda, |w|\le 2|z|} |E_2(z/w)| &= \sum_{k=1}^{[|z|]} \sum_{|w|\in[k,k+1)} \log\left|1-\frac{z}{w}\right| + \left|\frac{z}{w}\right| + \frac{1}{2}\left|\frac{z}{w}\right|^2 \\
&\le c'|z|^2 \sum_{k=1}^{[|z|]} \frac{1}{k^2}\cdot k \\
&\le c'r^2 \log r,
\end{aligned}$$

where we have used the counting estiamte $|\{|w| \in [k, k+1)\}| \le 32k$, $c', \tilde{C}$ are universal constant independent of r. Therefore, the order of σ is at most 2.

(b) when $z \ne 0$, $\sigma(z) \ne 0$. Therefore, $\log\sigma$ is locally defined.

$$\log\sigma = \log z + \sum_{j=1}^{\infty} \log\left(1-\frac{z}{\tau_j}\right) + \frac{z}{\tau_j} + \frac{1}{2}\left(\frac{z}{\tau_j}\right)^2.$$

Differentiating it, we conclude that

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{1}{z-\tau_j} + \frac{1}{\tau_j} + \frac{z}{\tau_j^2}.$$

Noted that

$$\begin{aligned}
\frac{1}{z-\tau} + \frac{1}{\tau} + \frac{z}{\tau^2} &= \frac{1}{\tau^2(z-\tau)}\left[\tau^2 + \tau(z-\tau) + z(z-\tau)\right] \\
&= \frac{z^2}{\tau^2(z-\tau)} = O(\tau^{-3}).
\end{aligned}$$

Therefore, the right hand side defines a meromorphic function. By identity theorem, the equality holds on $\mathbb{C}$ except the lattice point.

(c) Followed by direct differentiation. Formal differentiation is valid due to the convergence of C^0 (uniform locally) and cauchy integral formula.

Ch9 Q6 It immediately follows from differentiating the functional equation in Theorem 1.7 in textbook. Or we can argue by observing the behaviour around the poles just like theorem 1.7.

Ch9 Q7

$$\pi^2 = \sum_{m=-\infty}^{\infty} \frac{1}{(m+1/2)^2} = 8 \sum_{m\geq 1, m\ odd} \frac{1}{m^2}.$$

On the other hand,

$$\sum_{m=1}^{\infty} \frac{1}{(m+\tau)^2} + \sum_{m=-\infty}^{-1} \frac{1}{(m+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)} - \frac{1}{\tau^2}$$

Since, $\sin x = x - \frac{x^3}{6} + O(|x|^4)$,

$$\frac{\pi^2}{\sin^2(\pi\tau)} - \frac{1}{\tau^2} \approx \tau^{-2}\left(\frac{\pi^2\tau^2 - \sin^2(\pi\tau)}{\sin^2(\pi\tau)}\right) \to \frac{\pi^2}{3}, \quad \text{as } \tau \to 0.$$

Hence,

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

By differentiation, we have

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^4} = 3^{-1}\pi^4(\csc^2 \pi\tau)(\csc^2 \pi\tau + 2\cot^2 \pi\tau)$$

It remains to evaluate the following limit

$$L = \lim_{\tau\to 0} \frac{1}{3\sin^4(\pi\tau)}\left[\pi^4(1+2\cos^2(\pi\tau)\right] - \frac{1}{\tau^4}.$$

The rest will follow identically. By power series expansion,

$$\begin{aligned}
\left(\frac{\sin z}{z}\right)^4 &= \left(1 - \frac{z^2}{6} + \frac{z^4}{120} + ...\right)^4 \\
&= (1 - \frac{z^2}{6})^4 + 4(1 - \frac{z^2}{6})^3 \frac{z^4}{120} + ... \\
&= 1 - \frac{2}{3}z^2 + \frac{z^4}{5} + ...
\end{aligned}$$

And,

$$\cos^2 z = \left(1 - \frac{z^2}{2} + \frac{z^4}{24} + ...\right)^2 = 1 - z^2 + \frac{z^4}{3} + ...$$

Therefore,

$$L = \frac{\pi^4}{45}.$$